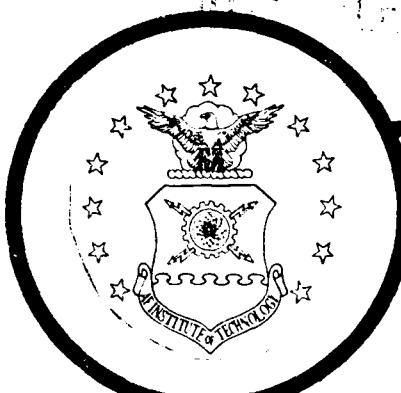


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BY HETEROGENEOUS FORCES

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AIR FORCE INSTITUTE OF TECHNOLOGY

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**MATHEMATICAL MODELING OF COMBAT ENGAGEMENTS
BY HETEROGENEOUS FORCES**

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ABSTRACT

The mathematical modeling of combat engagements by heterogeneous tactical forces is discussed and a new superiority parameter is introduced which represents a measure of the effectiveness of the opposing forces. This parameter is a function of both the quantitative and qualitative strengths of the opposing forces. It is defined as a product of the transpose of the left dominant eigenvector of the governing matrix of attrition coefficients and the force strength vector. This parameter is used to establish superiority criteria for combat engagements. The application of this new concept to heterogeneous forces is illustrated for the case of a "one-on-two" tactical engagement.

I. Introduction

The Lanchester equations for the heterogeneous forces engaged in a direct fire combat, can be represented symbolically by

$$\dot{X} = CX \quad (1)$$

where

$$X = (M \ N) \quad (2)$$

and C is the matrix of attrition coefficients. Here M and N are column

matrices denoting the force (weapon) numbers of the Blue and Red forces respectively. The braces are used for convenience throughout this report to denote column matrices. As discussed by Taylor¹, several authors have introduced aggregated force parameters to obtain a better insight into the effects of varying force size, the weapon allocation, and the weapon effective firing rate represented by Eq. (1). This approach involved essentially computation of two quantities: the aggregated force strengths F_B and F_R for the Blue and Red forces, respectively, which may be defined as

$$F_B = \bar{v}^T M \quad (3)$$

$$F_R = \bar{w}^T N \quad (4)$$

where \bar{v} and \bar{w} are positive vectors representing relative values of individual weapons while M and N are column matrices representing the force (weapon) numbers for the Blue and Red forces, respectively. Starting with the assumptions (3) and (4), a number of authors used the eigenvector method of computing \bar{v} and \bar{w} (see Section 3). In this method, however, both vectors could only be determined up to a constant multiplier. A simple scaling relationship was proposed by Dore and James² as $\sum \bar{v}_i = \sum \bar{w}_j = 1$. Other plausible scaling methods were introduced by different authors. For example, Taylor¹ in his book on Lanchester models of warfare, mentioned three other schemes³⁻⁶ for computing the scaling factors for the weapon value vectors \bar{v} and \bar{w} .

This report introduces a new way of determining the eigenvectors \bar{v} and \bar{w} through a mathematical representation of the complete analytical solution and the application of the dominant left eigenvector q_1 of the matrix C . This

approach leads also to a new measure of effectiveness $q_i^T X_0$ or its corresponding nondimensional superiority parameter S_0 , both of which determine a priori the direction of the outcome of the battle engagement without the necessity of solving the differential equations representing the engagement. The application of this new method of analysis is demonstrated here for the case of a tactical engagement involving three different weapons: one Blue and two Red weapons. In order to illustrate the nature of the analytical solution and, in particular, the presence of a dominant term in the solution which is used to derive the superiority parameter S_0 , a complete discussion of the general analytical solution is included in the Appendix.

2. Lanchester Equations for Heterogeneous Force Combat: "Many-on-Many" Combat Engagements.

The two separate governing equations for the heterogeneous force combat engagements, from Eq.(1), are given by

$$dM/dt = - BN \quad (5)$$

$$dN/dt = - AM \quad (6)$$

where M and N are the matrices representing the Blue and Red forces (or weapons), respectively. The mathematical model for such engagements is represented as the state variables M and N interacting with each other through the appropriate attrition coefficients A_{ij} and B_{ij} of A and B as shown in Fig. 1. Here the attrition coefficients take into account the firing rate effectiveness and the allocation of weapons. The matrices A and B are nonnegative, a property which will be used in the subsequent analysis. Also

$$M_i, N_i > 0, \text{ for all } i \quad (7)$$

and when this condition is not satisfied, i.e. when one of the forces is annihilated (when a particular M_i or $N_i = 0$), Eqs.(5) and (6) must be reconfigured by striking out the appropriate rows and columns.

Equations (5) and (6) can be combined into a single matrix equation

$$\frac{dX}{dt} = \dot{X} = CX \quad (1)$$

where

$$X = (M \ N) \quad (2)$$

and

$$C = \begin{bmatrix} 0 & -B \\ -A & 0 \end{bmatrix} \quad (8)$$

If the i th row in C contains all zeros this means that the corresponding $X_i = 0$ and the X_i -force is not being fired at. If the j th column in C contains all zeros then the X_j -force is being inactive.

The left handed eigenvector q of the matrix C and its corresponding eigenvalue λ are determined from

$$C^T q = \lambda q \quad (9)$$

The eigenvector q can be partitioned into rows corresponding to the Blue and Red forces, respectively, so that

$$q = (c_1 v \ c_2 w) \quad (10)$$

where v and w are the column vectors while c_1 and c_2 are constant multipliers.

It should be observed that Eq. (1) must represent a set of coupled equations. Uncoupled sets are not admissible in the present analysis because they represent unrelated (uncoupled) tactical engagements. Examples of uncoupled and coupled engagements are shown in Fig. 2, where arrows indicate the direction of fire. For these examples, the corresponding matrices of attrition coefficients are shown by Eqs. (11a) and (11b) where x's indicate the nonzero coefficients in C .

$$C = \begin{bmatrix} 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -x \\ -x & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \end{bmatrix}, \text{uncoupled sets} \quad (11a)$$

$$C = \begin{bmatrix} 0 & 0 & -x & x \\ 0 & 0 & 0 & -x \\ -x & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \end{bmatrix}, \text{coupled sets} \quad (11b)$$

The test whether the matrix C represents uncoupled sets of equations is simply whether it can be reduced to two or more nonzero diagonal submatrices by rearranging rows and columns in C , while all other off-diagonal submatrices are zero. Otherwise, the matrix C represents coupled sets of equations.

Substituting Eqs. (8) and (10) into Eq. (9),

$$\begin{bmatrix} 0 & -A^T \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} c_1 v \\ c_2 w \end{bmatrix} = \lambda \begin{bmatrix} c_1 v \\ c_2 w \end{bmatrix} \quad (12)$$

from which

$$-A^T c_2 w = \lambda c_1 v \quad (13)$$

$$-B^T c_1 v = \lambda c_2 w \quad (14)$$

Premultiplying Eq.(13) by B^T and then substituting Eq.(14) and premultiplying Eq.(14) by A^T and then substituting Eq.(13), the following pair of equations is obtained

$$(A^T B^T - \lambda^2 I) c_1 v = 0 \quad (15)$$

$$(B^T A^T - \lambda^2 I) c_2 w = 0 \quad (16)$$

If the dimensions of A are $n \times m$ and those of B are $m \times n$ and if $m > n$, then there will be n common eigenvalues $\lambda^2 = \mu$ in Eqs. (15) and (16). In addition, there will be $(m-n)$ eigenvalues of Eq. (15) equal to zero for which their corresponding eigenvectors can be obtained from

$$A^T B^T c_1 v = 0 \quad (17)$$

If those eigenvalues were not equal to zero, this would then imply that in addition to the $2n$ eigenvalues $\lambda = \pm \sqrt{\mu}$ in C there would be additional $2(m-n)$ eigenvalues for a total $2n+2(m-n) = 2m$ which would be greater than the required number $(m+n)$.

The matrix products $A^T B^T$ and $B^T A^T$ in Eqs. (15) and (16) are nonnegative since all A_{ij} and B_{ij} are either positive or zero. Therefore, according to the Frobenius-Perron theorem (Ref. 7, p. 193), Eqs. (15) and (16) have a real nonnegative eigenvalue λ^2 which is either equal to or exceeds the moduli of any other eigenvalue. Also to this maximal eigenvalue there corresponds an eigenvector with all nonnegative elements which will be denoted as $c_1 v$ and $c_2 w$ for Eqs. (15) and (16), respectively. Since $c_1 v$ and $c_2 w$ are submatrices of the left eigenvector of C it follows from Eq. (10) that

$$q_1 = (c_1 v \quad c_2 w) \quad (18)$$

which can be determined directly from Eq. (9). When computing \mathbf{v}_1 and \mathbf{w}_1 from Eqs. (15) and (16) any suitable scaling method can be used. For example the largest element can be made equal to unity, but the actual scaling between \mathbf{v}_1 and \mathbf{w}_1 is accomplished through the factors c_1 and c_2 to be determined later. Transposing Eqs. (13) and (14) with $\mathbf{v} = \mathbf{v}_1$ and $\mathbf{w} = \mathbf{w}_1$

$$- c_2 \mathbf{w}_1^T \mathbf{A} = \lambda_1 c_1 \mathbf{v}_1^T \quad (19)$$

$$- c_1 \mathbf{v}_1^T \mathbf{B} = \lambda_1 c_2 \mathbf{w}_1^T \quad (20)$$

Next, premultiplying the above equations by

$$\mathbf{e} = (1 \ 1 \ \dots \ 1) \quad (21)$$

and eliminating λ_1 the following relationship between c_1 and c_2 is obtained:

$$\begin{pmatrix} c_2 \\ c_1 \end{pmatrix}^2 = \begin{pmatrix} \mathbf{v}_1^T \mathbf{B} \mathbf{e} & \mathbf{v}_1^T \mathbf{e} \\ \frac{\mathbf{v}_1^T \mathbf{B} \mathbf{e}}{\mathbf{w}_1^T \mathbf{A} \mathbf{e}} & \frac{\mathbf{v}_1^T \mathbf{e}}{\mathbf{w}_1^T \mathbf{e}} \end{pmatrix} \quad (22)$$

$$\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \pm \begin{pmatrix} \mathbf{v}_1^T \mathbf{B} \mathbf{e} & \mathbf{v}_1^T \mathbf{e} \\ \frac{\mathbf{v}_1^T \mathbf{B} \mathbf{e}}{\mathbf{w}_1^T \mathbf{A} \mathbf{e}} & \frac{\mathbf{v}_1^T \mathbf{e}}{\mathbf{w}_1^T \mathbf{e}} \end{pmatrix}^{1/2} \quad (23)$$

The negative sign in Eq. (23) is selected to satisfy Eqs. (19) and (20) in which \mathbf{A} , \mathbf{B} , \mathbf{v}_1 , and \mathbf{w}_1 are all nonnegative matrices and λ_1 is a positive number. Coincidentally, Eq. (23) is of the same form suggested by Taylor¹ without proof for the so-called "summed result" in interpreting the aggregate force strengths. Equation (23) provides a rational relationship between \mathbf{v}_1

and \mathbf{w}_1 when these eigenvectors are calculated from Eqs. (15) and (16).

It should be noted that in practice the left dominant eigenvalue λ_1 and its corresponding eigenvector \mathbf{q}_1 can be obtained directly from Eq. (23) rather than from Eqs. (15), (16), and (23). Computer programs such as MATLAB are available which can be conveniently used to find the dominant eigenvalue λ_1 and the dominant left eigenvector \mathbf{q}_1 .

Because of the relationship (A.28), the generalized left eigenvectors \mathbf{q}_i^T are orthogonal to the right eigenvectors \mathbf{p}_j . This orthogonality property is expressed as

$$\begin{aligned} \mathbf{q}_i^T \mathbf{p}_j &= 1 \quad \text{for } i = j \\ &= 0 \quad \text{for } i \neq j \end{aligned} \quad (A.24)$$

This property is used next to derive a scalar parameter from the solution of the governing equations by premultiplying \mathbf{X} by the transpose of the dominant left eigenvector \mathbf{q}_1^T corresponding to the largest eigenvalue λ_1 . The resulting parameter is a measure of merit for the combat engagement. Consequently, employing Eqs. (A.25) and (A.14b) and noting that for the notation used in Appendix A $\lambda_1 = \lambda^{(1)}$ and $\mathbf{q}_1 = \mathbf{q}^{(1)}$

$$\begin{aligned} \left[\mathbf{q}_1^T \mathbf{X} \right]_{t=0} &= \left[\mathbf{q}_1^T \mathbf{P} \left[\exp(\lambda t) \mathcal{D} \right] \mathbf{Q}^T \mathbf{x}_0 \right]_{t=0} \\ &= \left[\mathbf{q}_1^T [\mathbf{p}^{(1)} \mathbf{p}^{(2)} \dots] \left[\exp(\lambda t) \mathcal{D} \right] \mathbf{Q}^T \mathbf{x}_0 \right]_{t=0} \end{aligned}$$

$$\begin{aligned}
&= \left[[1 \ 0 \ 0 \dots] \ 0 \ 0 \dots \right] \begin{bmatrix} \exp(\lambda_1^2 t) \lambda_1^{-1} & & & \\ & \exp(\lambda_2^2 t) \lambda_2^{-2} & & \\ & & \ddots & \\ & & & \exp(\lambda_n^2 t) \lambda_n^{-n} \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} X_O \\
&= \left[[\exp(\lambda_1^2 t) \ 1 \ 0 \ 0 \dots] \ 0 \ 0 \dots \right] (q_1^{(1)}^T \ q_2^{(2)}^T \ \dots \ q_n^{(n)}^T X_O)_{t=0} \\
&= \left[q_1^T \ X_O \right]_{t=0} = q_1^T X_O
\end{aligned} \tag{25}$$

Substituting now Eq. (18) for q_1 into Eq. (25) and observing that $X_O = (M_O \ N_O)$ it follows that

$$\begin{aligned}
q_1^T X_O &= [c_1 v_1^T \ c_2 w_1^T] \begin{bmatrix} M_O \\ N_O \end{bmatrix} \\
&= c_1 \left(v_1^T M_O + c_2 v_1^T w_1^T N_O \right) \\
&= c_1 \left(v_1^T M_O - \left(\frac{v_1^T B e}{w_1^T A e} - \frac{v_1^T e}{w_1^T e} \right)^{1/2} w_1^T N_O \right)
\end{aligned} \tag{26}$$

where for convenience c_1 can be taken as one. Because v_1 and w_1 are both nonnegative, the terms $v_1^T M_O$ and $w_1^T N_O$ are both positive. The scalar quantity $q_1^T X$ derived from the predominant term in the solution represents the difference between aggregate Blue and Red forces and can therefore be taken as a measure of merit, i.e., a superiority indicator for the opposing forces. It should be noted from the above that the analytical solution for X is a series of terms with multiplying factors $\exp(\lambda_i^2 t)$ which can be arranged in descending order for λ_i^2 as $\lambda_1^2, \lambda_2^2, \dots, 0, 0, 0, \dots, -\lambda_n^2, -\lambda_1^2$. Thus the term with λ_1^2 is the dominant term in the solution. If λ_1^2 is a repeated

eigenvalue, or if other eigenvalues are close to the dominant eigenvalue, the use of $q_1^T X_0$ as a measure of merit may not be appropriate. Consequently, it is important to check the distribution of eigenvalues before applying this measure of merit. In most practical cases, however, the $\lambda_i^{(1)}$'s are well separated. For example, for equal coefficients in both A and B (i.e., $A_{ij} = a$ and $B_{ij} = b$ for all i and j where a and b are constants), the eigenvalues are $\lambda^{(1)}, 0, 0, \dots, 0, 0, -\lambda^{(1)}$. $\lambda^{(1)} = \sqrt{mb/a}$ and $q_1 = (1 \quad 1 \quad \dots \quad m\text{-times} \quad -\sqrt{mb/a} \quad \dots \quad n\text{-times})$ where m and n refer to the number of Blue and Red forces, respectively (see Appendix). It is interesting to observe that for the multiple eigenvalue cases, the dominant eigenvector q_1 contains zero elements which means that some force strengths are not included in $q_1^T X_0$.

To reduce Eq. (26) into a nondimensional parameter, the equation can be normalized with respect to $c_1 v_1^T M_0$ by introducing a nondimensional superiority parameter S_0 such that

$$S = q_1^T X / c_1 v_1^T M_0 \quad (27)$$

$$\text{and } S_0 = \frac{q_1^T X_0}{c_1 v_1^T M_0} = \left(1 - \left(\frac{v_1^T B e}{w_1^T A e} \frac{v_1^T e}{w_1^T e} \right)^{1/2} \frac{w_1^T N_0}{v_1^T M_0} \right) \\ = 1 - F_0^{-1} \quad (28)$$

where

$$F_0 = \left(\frac{w_1^T A e}{v_1^T B e} \frac{w_1^T e}{v_1^T e} \right)^{1/2} \frac{v_1^T M_0}{w_1^T N_0} \quad (29)$$

The nondimensional parameters S, S_0 , and F_0 are all independent of the choice of the arbitrary multiplying factors with v_1 and w_1 . The superiority

parameter S is a scalar quantity measuring the contributions of the dominant term in the solution for the force levels at any given time. It should also be noted that

$$\begin{aligned} 0 < S_o < 1 & \quad \text{for } 1 < \Phi_o < \infty \\ S_o = 0 & \quad \text{for } \Phi_o = 1 \\ -\infty < S_o < 0 & \quad \text{for } 0 < \Phi_o < 1 \end{aligned} \quad (30)$$

The parameters $\mathbf{q}_1^T \mathbf{X}_o$ or S_o can be used as measures of effectiveness to determine the outcome of a particular tactical engagement without actually solving the differential equations subject to the restrictive conditions discussed earlier for the measure of merit $\mathbf{q}_1^T \mathbf{X}_o$. Thus as in the case of a "one-on-one" engagement the Blues forces win the engagement when $\Phi_o > 1$ or when $S_o > 0$, while the Red forces win when $\Phi_o < 1$ or when $S_o < 0$. When $\Phi_o = 1$ or $S_o = 0$, both sides reduce their strengths proportionately.

It should be noted that as in the case of a "one-on-one" engagement the normalized force ratio Φ_o for the general case consists of a product of two components representing the qualitative and quantitative ratios. The qualitative ratio is $[(\mathbf{w}_1^T \mathbf{Ae})(\mathbf{w}_1^T \mathbf{e}) / (\mathbf{v}_1^T \mathbf{Be})(\mathbf{v}_1^T \mathbf{e})]^{1/2}$ and the quantitative ratio is $(\mathbf{v}_1^T \mathbf{M}_o) / (\mathbf{w}_1^T \mathbf{N}_o)$.

3. Aggregated Force Strengths

To reduce the mathematical representation of the "many-on-many" engagements to a "one-on-one" representation the aggregated force strengths may be introduced into Eqs. (5) and (6). Premultiplying Eq. (5) by $c_1 \mathbf{v}_1^T$ and Eq. (6) by $c_2 \mathbf{w}_1^T$, the following differential equations are obtained:

$$\overset{\cdot}{c_1 \mathbf{v}_1^T} \mathbf{M} = -\overset{\cdot}{c_1 \mathbf{v}_1^T} \mathbf{B} \mathbf{N} = \beta c_2 \mathbf{w}_1^T \mathbf{N} \quad (31)$$

$$\overset{\cdot}{c_2 \mathbf{w}_1^T} \mathbf{N} = -\overset{\cdot}{c_2 \mathbf{w}_1^T} \mathbf{A} \mathbf{M} = \alpha c_1 \mathbf{v}_1^T \mathbf{M} \quad (32)$$

where the overdots represent differentiation with respect to time and the right sides of these equations, with positive constants α and β , have at this point been introduced arbitrarily. Introducing the aggregated force strengths F_B and F_R from Eqs. (3) and (4) as

$$F_B = \bar{\mathbf{v}}^T \mathbf{M} = c_1 \mathbf{v}_1^T \mathbf{M} \quad (33)$$

$$F_R = \bar{\mathbf{w}}^T \mathbf{N} = -c_2 \mathbf{w}_1^T \mathbf{N} \quad (34)$$

where

$$\bar{\mathbf{v}} = c_1 \mathbf{v}_1 \quad (35)$$

$$\bar{\mathbf{w}} = -c_2 \mathbf{w}_1 \quad (36)$$

Hence, substituting Eq. (33) and (34) into Eqs. (31) and (32)

$$\dot{F}_B = -\beta F_R \quad (37)$$

$$\dot{F}_R = -\alpha F_B \quad (38)$$

which are the governing equations for a "one-on-one" engagement between the aggregated forces F_B and F_R ; however, a word of caution is appropriate here. The solutions for F_B and F_R are only approximations to the exact solutions because they do not incorporate the condition that one of the M_i or N_i can be reduced to zero independently of others. In the aggregated solution of Eqs. (37) and (38) all components of F_B or F_R are reduced to zero simultaneously.

Since \mathbf{M} and \mathbf{N} are arbitrary, in order for the right sides of Eqs. (31)

and (32) to be valid the following relations must be true:

$$-\mathbf{c}_1 \mathbf{v}_1^T \mathbf{B} = \beta \mathbf{c}_2 \mathbf{w}_1^T \quad \text{or} \quad -\mathbf{B}^T \mathbf{c}_1 \mathbf{v}_1 = \beta \mathbf{c}_2 \mathbf{w}_1 \quad (39)$$

$$-\mathbf{c}_2 \mathbf{w}_1^T \mathbf{A} = \alpha \mathbf{c}_1 \mathbf{v}_1^T \quad \text{or} \quad -\mathbf{A}^T \mathbf{c}_2 \mathbf{w}_1 = \alpha \mathbf{c}_1 \mathbf{v}_1 \quad (40)$$

Solving for \mathbf{v}_1 and \mathbf{w}_1 ,

$$\begin{aligned} \mathbf{A}^T \mathbf{B}^T \mathbf{c}_1 \mathbf{v}_1 &= -\beta \mathbf{A}^T \mathbf{c}_2 \mathbf{w}_1 = \beta \alpha \mathbf{c}_1 \mathbf{v}_1 \\ (\mathbf{A}^T \mathbf{B}^T - \alpha \beta \mathbf{I}) \mathbf{c}_1 \mathbf{v}_1 &= 0 \end{aligned} \quad (41)$$

$$\begin{aligned} \mathbf{B}^T \mathbf{A}^T \mathbf{c}_2 \mathbf{w}_1 &= -\alpha \mathbf{B}^T \mathbf{c}_1 \mathbf{v}_1 = \alpha \beta \mathbf{c}_2 \mathbf{w}_1 \\ (\mathbf{B}^T \mathbf{A}^T - \alpha \beta \mathbf{I}) \mathbf{c}_2 \mathbf{w}_1 &= 0 \end{aligned} \quad (42)$$

Hence, comparing Eqs. (41) and (42) with Eqs. (15) and (16)

$$\lambda_1^2 = \alpha \beta \quad (43)$$

Similarly, by comparing Eqs. (39) and (20) and Eqs. (40) and (19)

$$\alpha = \beta = \lambda_1 \quad (44)$$

which is the same result derived from the scaling method assumed by Holter⁵ and Anderson.⁶

The above analysis has demonstrated that a unified theory for the analysis of Lanchester equations for heterogeneous forces engaged in a direct fire combat can be developed. This is accomplished by correlating the quantities \mathbf{v}_1 , \mathbf{w}_1 , α and β , used previously by other authors, to the left

dominant eigenvector of the governing matrix of attrition coefficients and its dominant eigenvalue. The theory provides for a proper scaling of the eigenvectors v_1 and w_1 , and leads directly to the measure of effectiveness $q_1^T X_0$ or the superiority parameter S_0 . Although the use of the left dominant eigenvector introduced here for the combat analysis is new, similar approach has been used in the past in other areas. For example the use of the left dominant eigenvector in the cohort population model that leads to the total reproductive value of the population as described by Luenberger (Ref.7, p.185).

4. Numerical Example: "One-On-Two" Tactical Engagement

The proposed method of analysis of heterogeneous forces combat is illustrated here for the case of a tactical engagement of one Blue force and two Red forces as shown in Fig. 3. For this example the equations describing the engagement have been assumed as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -3/2 & -1 \\ -4/3 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

It can be verified easily that for this case

$$X = (x_1 \ x_2 \ x_3) = (M_1 \ N_1 \ N_2)$$

$$q_1 = (1 \ -3/4 \ -1/2)$$

$$v_1^T = [1] ; \quad w_1^T = [1 \ 2/3]$$

$$\alpha = f^2 = \lambda_1 = 2$$

and

$$\Phi_0 = \frac{4}{3} \frac{x_1}{(x_2 + (2/3)x_3)}$$

In Figs. 4, 5 and 6 the variation of x_1 , x_2 , and x_3 with the nondimensional time τ is shown for $X_0 = (200 100 100)$, $(125 100 100)$, and $(100 100 100)$ which correspond to $\Phi_0 = 1.6$, 1.0 , and 0.8 , respectively. The solutions for X were obtained by numerical integration. For Case 1 (Fig. 4) in which $\Phi_0 = 1.6$, $x_3 = 0$ when $\tau = 0.57$ and at that time the portion of x_1 allocated to fight x_3 is reallocated against x_2 , requiring that the equations must be reformulated. Subsequently the x_2 -force is reduced quickly to zero by the x_1 -force. As the battle progresses, the superiority parameter S shows a rapid increase (see Fig. 7). Here the initial positive value of S is an indication of the superiority of the Blue force. For Case 2 in which $\Phi_0 = 1.0$, the opposing forces diminish their strengths gradually without any side gaining a clear advantage (see Fig. 5). Here $S = 0$. Case 3 for which $\Phi_0 = 0.8$ represents a rapid demise of the x_1 -force (see Fig. 6). For this case S starts with a negative value which diminishes as the battle progresses (see Fig. 7). The initial negative value of S is an indication of the inferiority of the Blue force.

Since the dominant left eigenvector is the key parameter the proposed measure of merit S_0 , the eigenvectors q_1 are shown as examples in Appendix B for the cases of "m-on-n" engagements for which all attrition coefficients are equal.

5. Conclusions

The premultiplication of the solution vector $X = (M N)$ for the engaging forces by the transpose of the dominant left eigenvector q_1 of the matrix of attrition coefficients helps to clarify the meaning of the eigenvector methods.

of determining the relative values of heterogeneous forces in a combat engagement. Although, the eigenvector methods have been around for a long time, problems existed in deciding on the best method of scaling the resulting eigenvectors. The present report demonstrates clearly the meaning of the individual eigenvectors v_1 and w_1 as components of the dominant left eigenvector of the governing matrix of attrition coefficients. This in turn, allows for a proper determination of the relative scaling of eigenvectors for the two opposing forces and it leads to the nondimensional superiority parameter S_0 which can be used as a measure of merit and an indicator of the final outcome of the battle engagement.

The practical utility of the proposed measure of merit can be demonstrated by obtaining numerical solutions to heterogeneous force engagements while at the same time computing the parameter S introduced in this paper. The sign of the parameter S and the extent to which it is either increasing or decreasing can be used as a measure of superiority of the Blue forces in relation to the Red forces. Although the method is valid only for constant attrition coefficients, it could also be of some value for cases with variable coefficients where the instantaneous values of S could be used to indicate the general trends as the battle engagement progresses.

APPENDIX

Analytical Solution of Lanchester Equations

a. General Solution of $\dot{X} = CX$

The general solution of the fundamental equation

$$\dot{X} = CX \quad (A1)$$

can be obtained as a product solution of the form

$$X = P\varphi \quad (A2)$$

where P is the matrix of *generalized* right eigenvectors and φ is the column matrix whose elements are function of time. Substituting Eq.(A2) into (A1)

$$\dot{X} = P\dot{\varphi} = CP\varphi$$

Hence

$$\dot{\varphi} = P^{-1}CP\varphi = J\varphi \quad (A3)$$

where

$$\varphi = (\varphi^{(1)} \varphi^{(2)} \dots \varphi^{(n)} \dots) \quad (A4)$$

$$\dot{\varphi} = (\dot{\varphi}^{(1)} \dot{\varphi}^{(2)} \dots \dot{\varphi}^{(n)} \dots) \quad (A5)$$

and

$$J = P^{-1}CP \quad (A6a)$$

$$= \begin{bmatrix} J^{(1)} & & & \\ & J^{(2)} & & 0 \\ & & \ddots & \\ 0 & & & J^{(n)} \end{bmatrix} \quad (A6b)$$

is the Jordan canonical form of the matrix C , modified to take into the account the dimensionality of the matrix elements. The governing equation matrix C , its eigenvalues, and the matrix J are of the dimension 1/time. Also for reasons explained later J is taken here as the lower triangular matrix instead of the conventional upper triangular matrix.

A typical term in Eq.(A3) is of the form

$$\dot{\varphi}^{(i)} = J^{(i)}\varphi^{(i)} \quad (A7)$$

$$\dot{\varphi}^{(i)} = (\dot{\varphi}_1^{(i)} \dot{\varphi}_2^{(i)} \dots \dot{\varphi}_k^{(i)})_{k:1} \quad (A8)$$

$$\varphi^{(i)} = \{ \varphi_1^{(i)}, \varphi_2^{(i)}, \dots, \varphi_k^{(i)} \}_{k \geq 1} \quad (\text{A9})$$

where $J^{(i)}$ is essentially one of the three types, or combinations thereof, as discussed below.⁸

Type (i):

$$J^{(i)} = \lambda^{(i)} I \quad (\text{A10})$$

when the algebraic and geometric multiplicity of the eigenvalue $\lambda^{(i)}$ is equal to one.

Type (ii):

$$J^{(i)} = \lambda^{(i)} I \quad (\text{A11})$$

when the algebraic and geometric multiplicity of the eigenvalue $\lambda^{(i)}$ is equal to k with $k \geq 2$ and the corresponding eigenvectors are linearly independent.

Type (iii):

$$J^{(i)} = \lambda^{(i)} \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}_{k \times k} \quad (\text{A12})$$

when the algebraic multiplicity of the eigenvalue $\lambda^{(i)}$ exceeds its geometric multiplicity. The expression for $J^{(i)}$ has been modified here from its standard textbook form in which only the diagonal terms are equal to $\lambda^{(i)}$, to ensure proper dimensionality of all its elements. The eigenvalues $\lambda^{(i)}$ in two different $J^{(i)}$'s can be the same. Also the Type (iii) may appear in combination with Types (i) and (ii). When $k=1$ the Type (iii) reduces to Type (i).

It is easy to verify that the solution to Eq.(A7) with $J^{(i)}$ given by Eq.(A12) (i.e. Type (iii)) can be obtained by solving first for $\varphi_1^{(i)}$ and then substituting it into the equation for $\varphi_2^{(i)}$. This process is repeated until the solution for $\varphi_k^{(i)}$ is found. These solutions can be written concisely as

$$\varphi^{(i)} = \exp(\lambda^{(i)} t) \left[I + \frac{\lambda^{(i)} t}{1!} H_1 + \frac{(\lambda^{(i)} t)^2}{2!} H_2 + \dots \right] A^{(i)} \quad (A13a)$$

$$= \exp(\lambda^{(i)} t) \left[I + \frac{\lambda^{(i)} t}{1!} H + \frac{(\lambda^{(i)} t)^2}{2!} H^2 + \dots \right] A^{(i)} \quad (A13b)$$

$$= \exp(\lambda^{(i)} t) D^{(i)} A^{(i)} \quad (A13c)$$

where

$$D^{(i)} = I + \frac{\lambda^{(i)} t}{1!} H + \frac{(\lambda^{(i)} t)^2}{2!} H^2 + \dots + \frac{(\lambda^{(i)} t)^{k-1}}{(k-1)!} H^{k-1} \quad (A14a)$$

$$= \begin{bmatrix} 1 & & & & & \\ \lambda^{(i)} t & 1 & & & & 0 \\ (\lambda^{(i)} t)^2/2 & \lambda^{(i)} t & 1 & & & \\ (\lambda^{(i)} t)^3/6 & (\lambda^{(i)} t)^2/2 & \lambda^{(i)} t & 1 & & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (A14b)$$

and

$$A^{(i)} = (A_1^{(i)} \ A_2^{(i)} \ \dots \ A_k^{(i)}) \quad (A15)$$

is the column matrix of constants of integration. The H-matrices are given by

$$H_1 = H = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & 0 \\ 0 & 1 & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}_{k \times k} \quad (A16)$$

$$H_2 = H^2 = \begin{bmatrix} 0 & & & & \\ 0 & 0 & & & 0 \\ 1 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}_{k \times k} \quad (A17)$$

etc.

which finally leads to

$$H^k = 0 \quad (A18)$$

Substituting Eq.(A13c) into (A4) and (A2), it follows that the general solution of $\dot{X} = CX$ can be written as

$$X = P\varphi = P \left[\exp(\lambda t)D \right] \mathcal{A} \quad (A19)$$

where

$$\left[\exp(\lambda t)D \right] = \left[\exp(\lambda^{(1)}t)D^{(1)} \exp(\lambda^{(2)}t)D^{(2)} \dots \right] \quad (A20)$$

$$\mathcal{A} = \{ A^{(1)} A^{(2)} \dots A^{(r)} \dots \} \quad (A21)$$

At time $t=0$, $X=X_0$ and therefore from Eq.(A19)

$$X_0 = PA \quad (A22)$$

since $\exp(0)=1$ and $D=I$ for $t=0$. Hence

$$A = P^{-1}X_0 = Q^T X_0 \quad (A23)$$

where the matrix Q is defined as

$$Q = (P^{-1})^T \quad (A24)$$

Hence from Eqs.(A21) and (A24)

$$X = P \left[\exp(\lambda t)D \right] Q^T X_0 \quad (A25)$$

If the matrix C is not defective, i.e. its geometric multiplicity is equal to the algebraic multiplicity for all eigenvalues $\lambda^{(i)}$, then $D^{(i)}=I$ and the solution for X simplifies to

$$X = P \left[\exp(\lambda t) \right] Q^T X_0 \quad (A26)$$

b. Right and Left Generalized Eigenvectors

From Eqs.(A6) and (A24) it follows that

$$CP = PJ \quad (A27)$$

$$Q^T P = I \quad (A28)$$

$$PQ^T = J \quad (A29)$$

and

$$\begin{aligned} \mathbf{Q}^T \mathbf{C} &= \mathbf{P}^{-1} \mathbf{C} \\ &= \mathbf{P}^{-1} \mathbf{C} \mathbf{P} \mathbf{P}^{-1} \\ &= \mathbf{J} \mathbf{Q}^T \end{aligned} \quad (\text{A30})$$

Equation (A30) implies that \mathbf{Q} is the matrix of the generalized left eigenvectors, while Eqs.(A28) and (A29) indicate orthogonality between the right and left generalized eigenvectors.

c. Computation of Generalized Right and Left Eigenvectors

Introducing the individual generalized eigenvectors into \mathbf{P} and \mathbf{Q} such that

$$\mathbf{P} = [\mathbf{p}^{(1)} \mathbf{p}^{(2)} \dots \mathbf{p}^{(k)} \dots] \quad (\text{A31})$$

$$\text{and} \quad \mathbf{Q} = [\mathbf{q}^{(1)} \mathbf{q}^{(2)} \dots \mathbf{q}^{(k)} \dots] \quad (\text{A32})$$

it can be demonstrated that the application of Eqs.(A27) and (A30) for $\mathbf{J}^{(1)}$ of the Type C(iii) and order $k \times k$ leads to the following relations:

$$\begin{aligned} (\mathbf{C} - \lambda \mathbf{I}) \mathbf{p}_1 &= \lambda \mathbf{p}_2 & \text{or} & \quad (\mathbf{C} - \lambda \mathbf{I})^k \mathbf{p}_1 = \mathbf{0} \\ (\mathbf{C} - \lambda \mathbf{I}) \mathbf{p}_2 &= \lambda \mathbf{p}_3 & \text{or} & \quad (\mathbf{C} - \lambda \mathbf{I})^{k-1} \mathbf{p}_2 = \mathbf{0} \\ \vdots & \vdots & & \vdots \\ (\mathbf{C} - \lambda \mathbf{I}) \mathbf{p}_{k-1} &= \lambda \mathbf{p}_k & \text{or} & \quad (\mathbf{C} - \lambda \mathbf{I})^2 \mathbf{p}_{k-1} = \mathbf{0} \\ (\mathbf{C} - \lambda \mathbf{I}) \mathbf{p}_k &= \mathbf{0} & \text{or} & \quad (\mathbf{C} - \lambda \mathbf{I}) \mathbf{p}_k = \mathbf{0} \end{aligned} \quad (\text{A33})$$

and

$$\begin{aligned} \mathbf{q}_1^T (\mathbf{C} - \lambda \mathbf{I}) &= \mathbf{0} & \text{or} & \quad \mathbf{q}_1^T (\mathbf{C} - \lambda \mathbf{I}) = \mathbf{0} \\ \mathbf{q}_2^T (\mathbf{C} - \lambda \mathbf{I}) &= \lambda \mathbf{q}_1^T & \text{or} & \quad \mathbf{q}_2^T (\mathbf{C} - \lambda \mathbf{I})^2 = \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{q}_{k-1}^T (\mathbf{C} - \lambda \mathbf{I}) &= \lambda \mathbf{q}_{k-2}^T & \text{or} & \quad \mathbf{q}_{k-1}^T (\mathbf{C} - \lambda \mathbf{I})^{k-1} = \mathbf{0} \\ \mathbf{q}_k^T (\mathbf{C} - \lambda \mathbf{I}) &= \lambda \mathbf{q}_{k-1}^T & \text{or} & \quad \mathbf{q}_k^T (\mathbf{C} - \lambda \mathbf{I})^k = \mathbf{0} \end{aligned} \quad (\text{A34})$$

where for simplicity all superscripts (1) have been omitted and the subscripts

i through k refer to the eigenvectors comprising the columns in p^T and q^T . Since the analysis of the relative strengths in combat engagements involves the calculation of the dominant left eigenvector q_1 , it is clear that the selection of J as the lower triangular matrix was more convenient because in the case of defective matrices C it resulted in a simple expression

$$q_1^T (C - \lambda I) = 0$$

or

$$(C^T - \lambda I) q_1 = 0 \quad (6.35)$$

for the calculation of q_1 .

d. Examples of the Dominant Left Eigenvectors

The dominant left eigenvectors are illustrated here for several combat engagements for which all attrition coefficients are equal to a constant a . The number of engaging types of weapons is indicated by the subscripts (m,n) with each matrix C. All eigenvalues of C are shown, with the dominant eigenvalue(s) listed first. The dominant eigenvector q_1 has been normalized on its largest element for the Blue forces.

$$C = \begin{bmatrix} 0 & -b \\ -a & 0 \end{bmatrix}_{2,2} ; \quad \lambda = \sqrt{ab}, -\sqrt{ab}; \quad q_1 = \begin{pmatrix} 1 + \sqrt{b/a} \\ 1 \end{pmatrix}_{2 \times 1}$$

$$C = \begin{bmatrix} 0 & 0 & -b \\ 0 & 0 & -b \\ -a & -a & 0 \end{bmatrix}_{3,3} ; \quad \lambda = \sqrt{2ab}, 0, -\sqrt{2ab}; \quad q_1 = \begin{pmatrix} 1 + \sqrt{2b/a} \\ 1 \\ 1 \end{pmatrix}_{3 \times 1}$$

$$C = \begin{bmatrix} 0 & 0 & \dots & -b \\ 0 & 0 & \dots & -b \\ \vdots & \vdots & \ddots & \vdots \\ -a & -a & \dots & 0 \end{bmatrix}_{n,n} ; \quad \lambda = \sqrt{mb}, 0, \dots, 0, -\sqrt{mb}$$

$$q_1 = \begin{pmatrix} 1 + \sqrt{b/a} \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n+1 \times 1}$$

$$C = \begin{bmatrix} 0 & 0 & -b & -b \\ 0 & 0 & -b & -b \\ -a & -a & 0 & 0 \\ -a & -a & 0 & 0 \end{bmatrix}_{(2,2)} ; \lambda_1 = 2\sqrt{ab}, 0, 0, -2\sqrt{ab}$$

$$q_1 = \begin{pmatrix} 1 & 1 & -\sqrt{b/a} & -\sqrt{b/a} \end{pmatrix}_{4 \times 1}$$

$$C = \begin{bmatrix} 0 & 0 & -b & -b \\ 0 & 0 & -b & -b \\ 0 & 0 & -b & -b \\ -a & -a & -a & 0 \\ -a & -a & -a & 0 \end{bmatrix}_{(3,2)} ; \lambda_1 = \sqrt{6ab}, 0, 0, 0, -\sqrt{6ab}$$

$$q_1 = \begin{pmatrix} 1 & 1 & 1 & -\sqrt{3b/2a} & -\sqrt{3b/2a} \end{pmatrix}_{5 \times 1}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & -b & -b & -b \\ 0 & 0 & 0 & -b & -b & -b \\ 0 & 0 & 0 & -b & -b & -b \\ -a & -a & -a & 0 & 0 & 0 \\ -a & -a & -a & 0 & 0 & 0 \\ -a & -a & -a & 0 & 0 & 0 \end{bmatrix}_{(3,3)} ; \lambda_1 = 3\sqrt{ab}, 0, 0, 0, 0, -3\sqrt{ab}$$

$$q_1 = \begin{pmatrix} 1 & 1 & 1 & -\sqrt{b/a} & -\sqrt{b/a} & -\sqrt{b/a} \end{pmatrix}_{6 \times 1}$$

For the general case of m Blue forces on n Red forces (mean) with equal coefficients in \mathbf{A} and \mathbf{B}

$$\lambda_1 = \lambda^{(1)}, 0, 0, \dots, 0, 0, -\lambda^{(1)}$$

$$\lambda^{(1)} = \sqrt{mnab}$$

$$q_1 = \begin{pmatrix} 1 & 1 & \dots & m \text{ times } \sqrt{ab/mn} & \sqrt{mb/mn} & \dots & n \text{ times } \sqrt{ab/mn} \end{pmatrix}_{m+n+1 \times 1}$$

The following two examples with some zero attrition coefficients represent cases of defective matrices C with repeated eigenvalues.

$$C = \begin{bmatrix} 0 & 0 & -b & -b \\ 0 & 0 & 0 & -b \\ -a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \end{bmatrix}_{(2,2)} ; \lambda_1 = \sqrt{ab}, \sqrt{ab}, -\sqrt{ab}, -\sqrt{ab}$$

$$q_1 = \begin{pmatrix} 0 & 1 & 0 & -\sqrt{b/a} \end{pmatrix}_{4 \times 1}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & -b & -b & -b \\ 0 & 0 & 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 & 0 & -b \\ -a & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 \end{bmatrix}_{(3,3)} : \lambda = \sqrt{ab}, \sqrt{ab}, 0, 0, -\sqrt{ab}, -\sqrt{ab}$$

$$q_1 = (0 \ 0 \ 1 \ 0 \ 0 \ -\sqrt{b/a})_{6 \times 1}$$

e. Computation of the Dominant Left Eigenvectors and their Eigenvalues

In addition to the dominant eigenvalue λ_1 , the matrix C^T contains also an eigenvalue equal to $-\lambda_1$. Consequently, because there are two eigenvalues with equal absolute values the standard power method of computing the largest eigenvalue and its corresponding eigenvector can not be applied; however, the power method with a shift of origin can be used to find both λ_1 and q_1 . Starting with Eq. (C.35) for $\lambda=\lambda_1$,

$$(C^T - \lambda_1 I) = 0 \quad (C.35)$$

and adding μI to both sides, where μ is a constant (e.g., $\mu=1$),

$$(C^T + \mu I) q_1 = (\lambda_1 + \mu) q_1 \quad (C.36)$$

Hence,

$$\bar{C}^T q_1 = \tilde{\lambda} q_1 \quad (C.37)$$

where

$$\bar{C}^T = C + \mu I \quad (C.38)$$

$$\tilde{\lambda}_1 = \lambda_1 + \mu \quad (C.39)$$

In this way the shift of the origin for the eigenvalues λ_1 and $-\lambda_1$ has resulted in $|\tilde{\lambda}_1 + \mu|$ being greater than $|\lambda_1 + \mu|$; therefore, the standard power method can be applied to Eq. (C.38) to find q_1 and $\tilde{\lambda}_1$ and then λ_1 from Eq. (C.40).

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FIGURES

Fig. 1 Model of heterogeneous force engagements (Many-on-many).

Fig. 2 Example of uncoupled and coupled tactical engagements.

Fig. 3 Numerical example: one-on-two engagement.

Fig. 4 Numerical example (Case 1): Variation of force strengths X_1 , X_2 , X_3 ,
with the nondimensional time $\tau = \lambda^{(1)}t$ for $\Phi_0 = 1.6$ and $X_0 = (200 \ 100 \ 100)$.

Fig. 5 Numerical example (Case 2): Variation of force strengths X_1 , X_2 , X_3 ,
with the nondimensional time $\tau = \lambda^{(1)}t$ for $\Phi_0 = 1.0$ and $X_0 = (125 \ 100 \ 100)$.

Fig. 6 Numerical example (Case 3): Variation of force strengths X_1 , X_2 , X_3 ,
with the nondimensional time $\tau = \lambda^{(1)}t$ for $\Phi_0 = 0.8$ and $X_0 = (100 \ 100 \ 100)$.

Fig. 7 Numerical examples (Cases 1, 2, and 3): Variation of the superiority
parameter S with the nondimensional time $\tau = \lambda^{(1)}t$ for $\Phi_0 = 1.6$, 1.0, and
0.8.

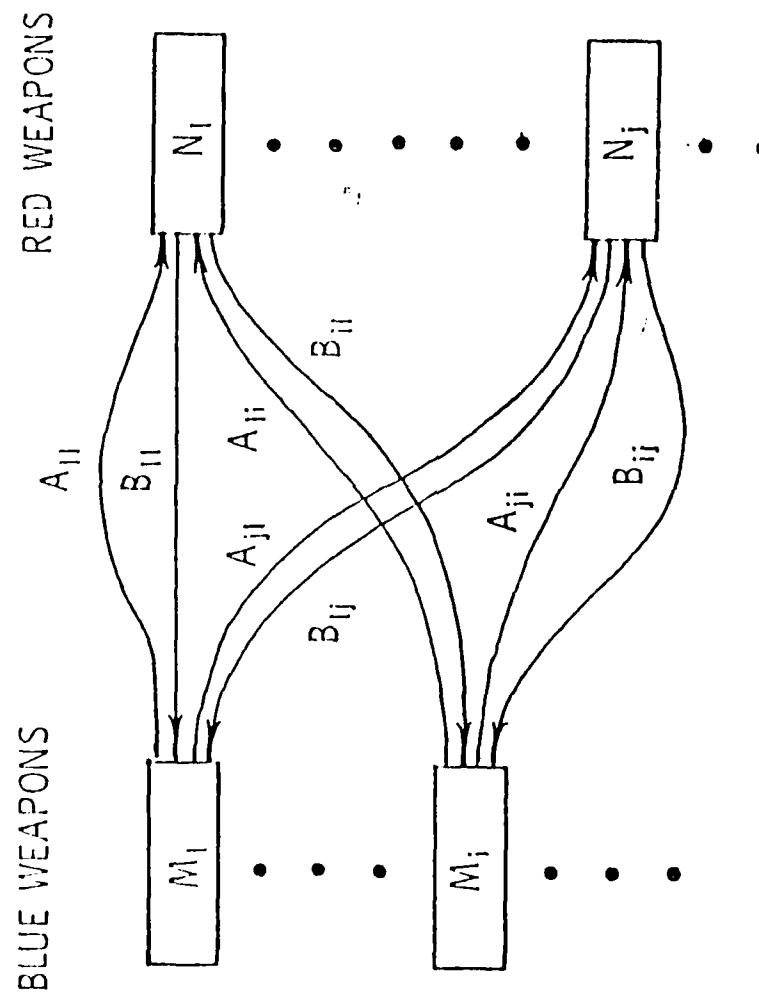
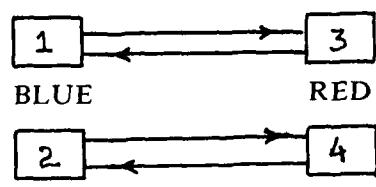
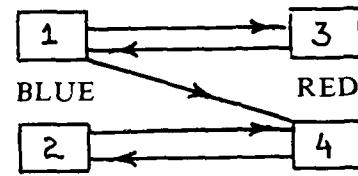


Fig. 1 Model of heterogeneous force engagements (Many-on-many).



a) Uncoupled engagements



b) Coupled engagements

Fig. 2 Examples of uncoupled and coupled tactical engagements.

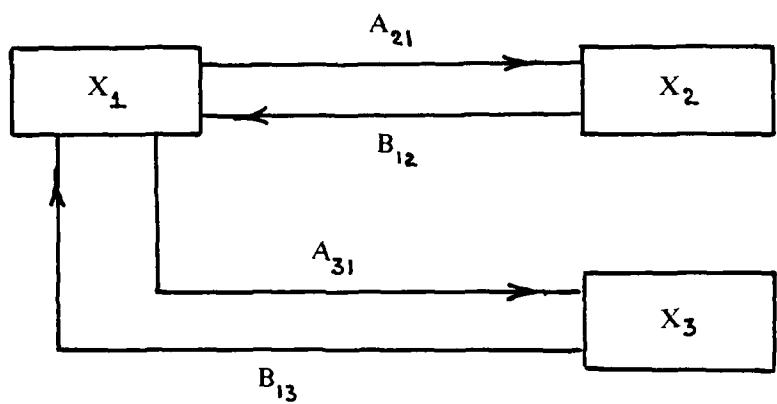


Fig. 3 Numerical example: one-on-two engagement.

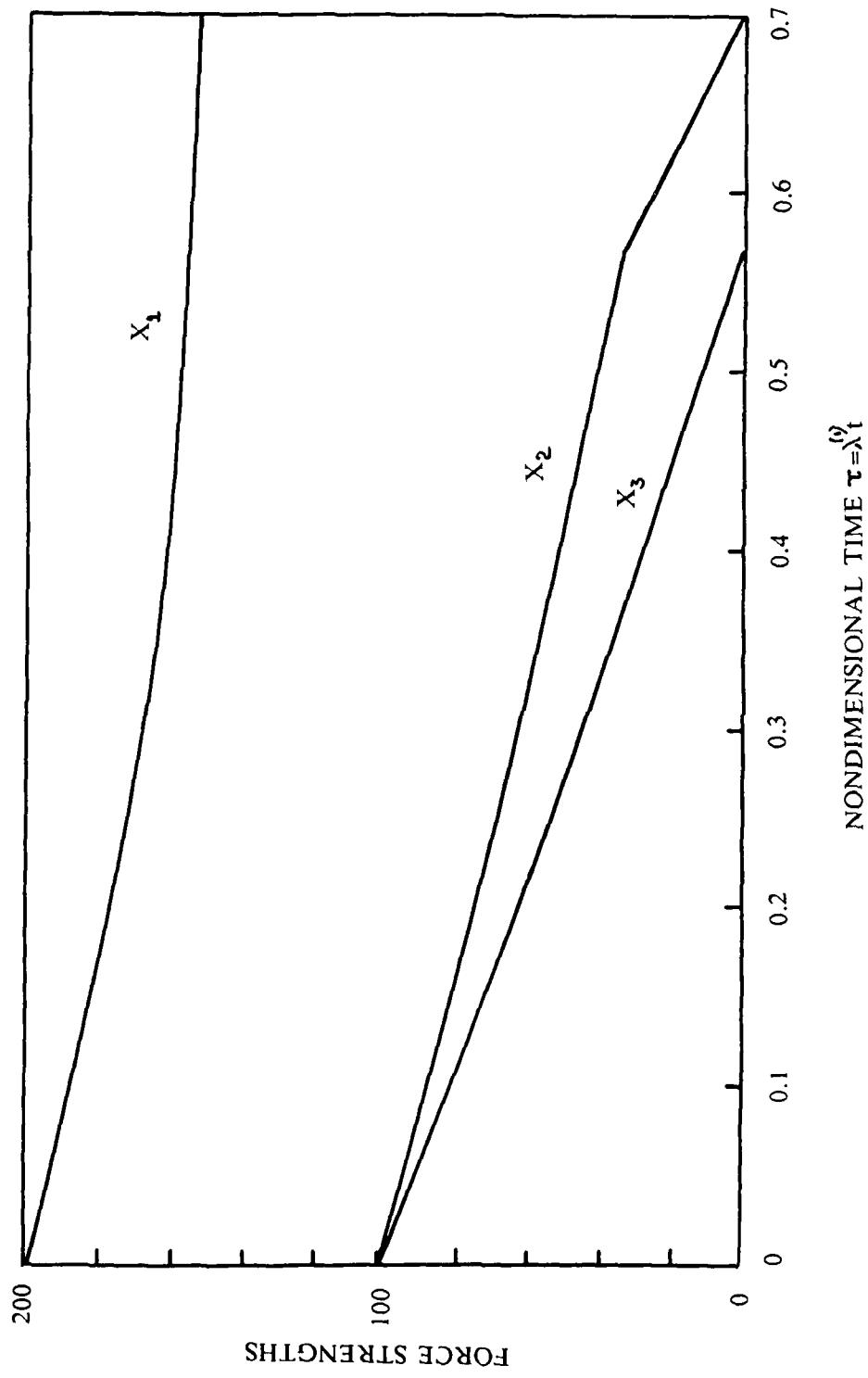


Fig. 4 Numerical example (Case 1): variation of the force strengths X_1 , X_2 , and X_3 , with the nondimensional time $\tau = \lambda t$ for $\Phi_o = 1.6$ and $X_0 = \{200 \ 100 \ 100\}$.

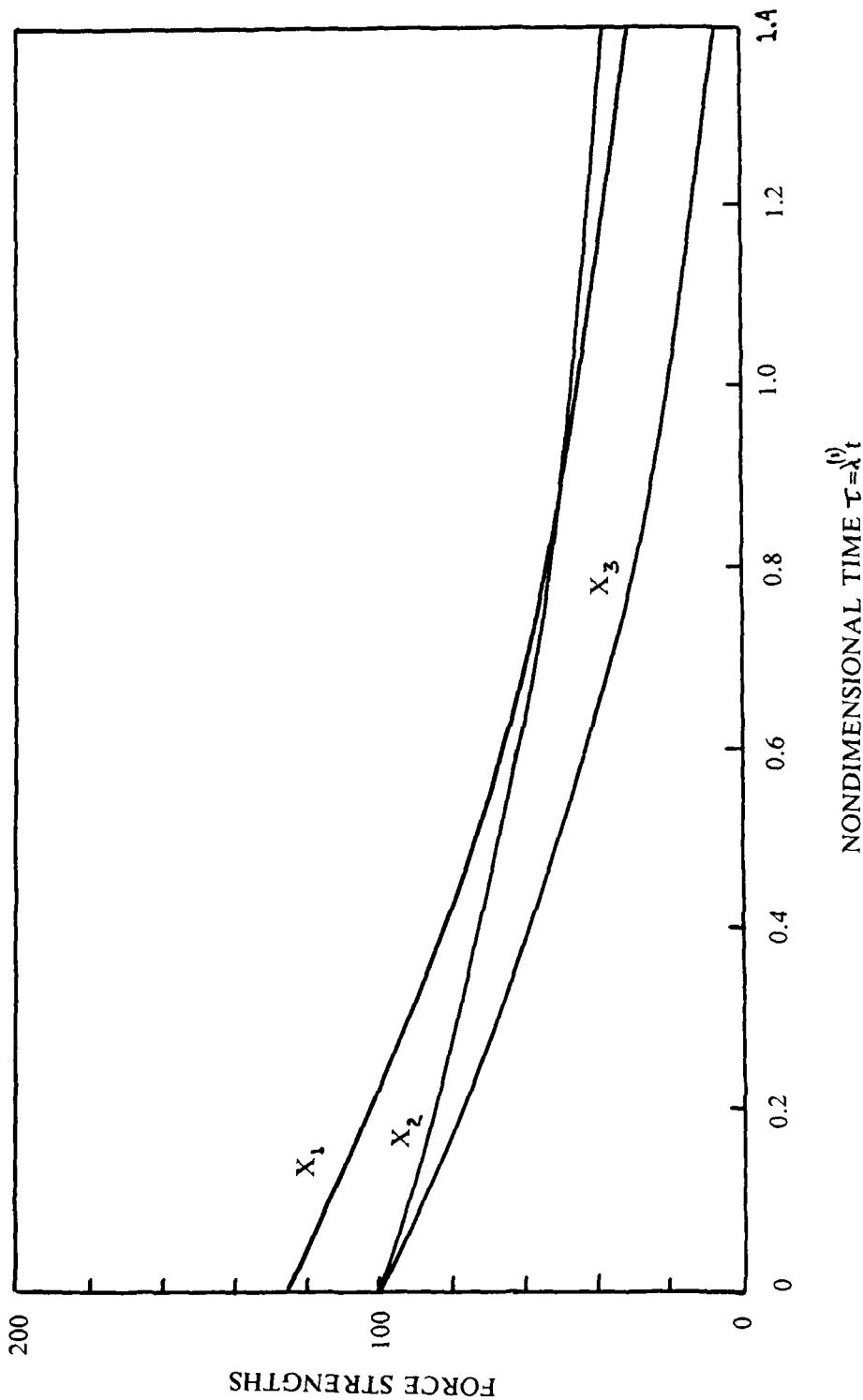


Fig. 5 Numerical example (Case 2): variation of the force strengths X_1 , X_2 , and X_3 , with the nondimensional time $\tau = \lambda^0 t$ for $\Phi_0 = 1.0$ and $X_0 = \{125, 100, 100\}$.

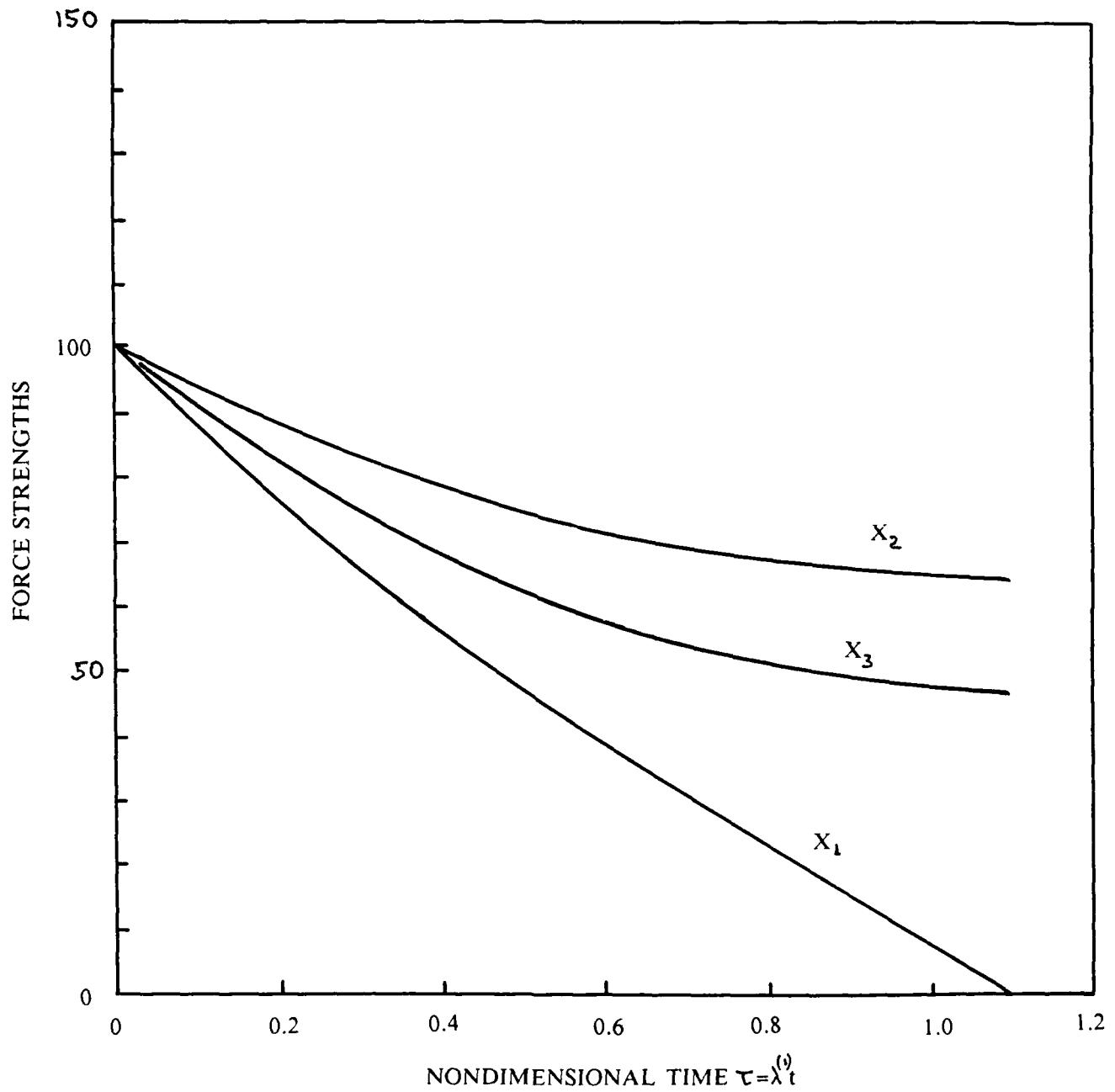


Fig. 6 Numerical example (Case 3): variation of the force strengths X_1 , X_2 , and X_3 , with the nondimensional time $\tau = \lambda^0 t$ for $\Phi_0 = 0.8$ and $X_0 = (100 \ 100 \ 100)$.

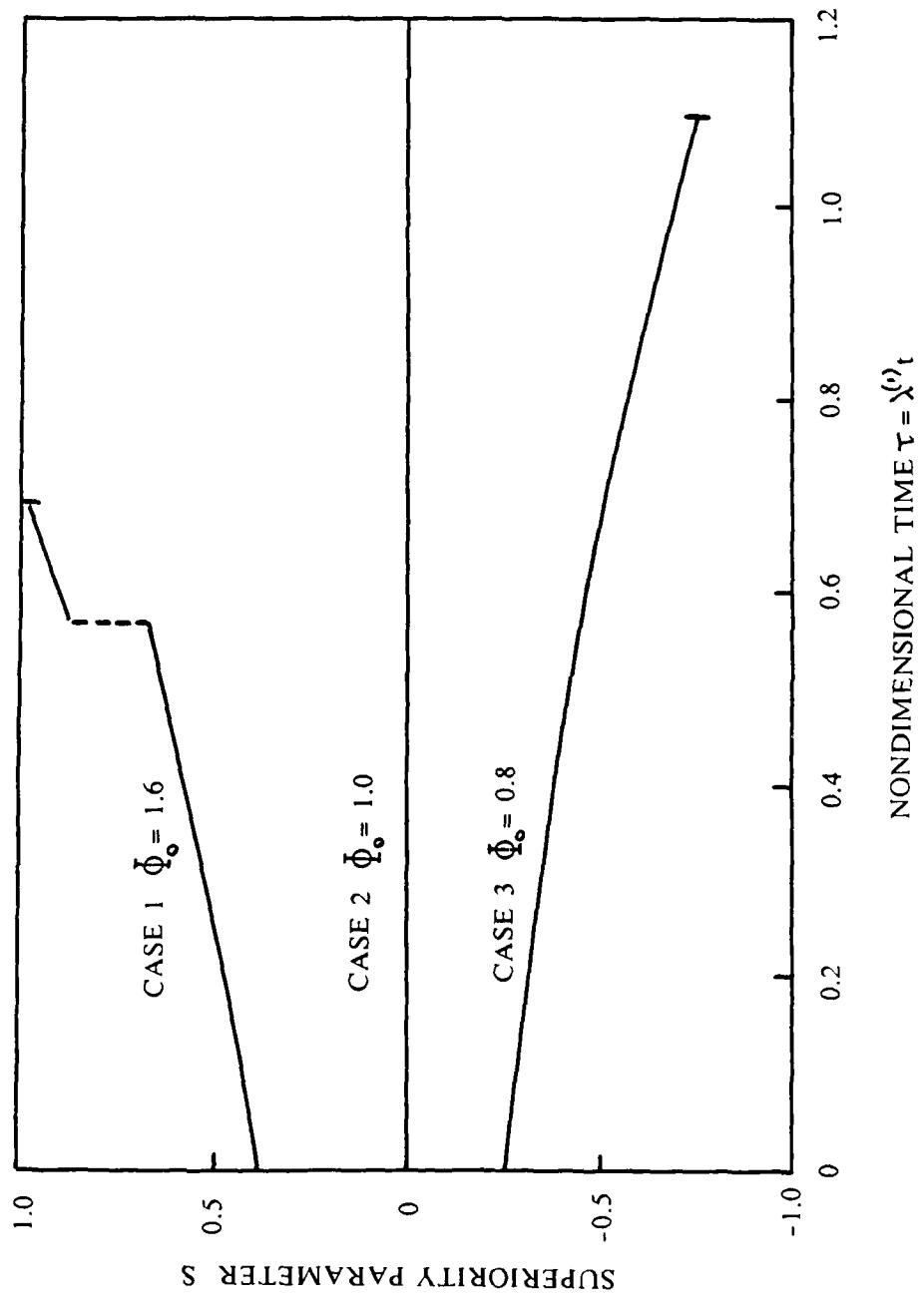


Fig. 7 Numerical examples (Cases 1, 2, and 3): variation of the superiority parameter S with the nondimensional time $\tau = X^0_t$ for $\Phi_0 = 1.6$, 1.0 and 0.8.

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